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PROPERTIES OF A DIFFERENTIAL GAME'S POTENTIAL*

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A position differential game with fixed termination instant is examined. Stability properties are investigated, consisting mainly in the formulation of necessary and sufficient conditions satisfied by the differential game's value function (potential). The infinitesimal form of the stability properties leads to differential inequalities generalizing the fundamental equation for the potential to the case when the value function is nondifferentiable. As a corollary to the necessary and sufficient conditions obtained for the differential game's potential, corresponding results are presented for the optimal control problem. The paper relies on the results from /1 – 7/, borders on the studies in /8 – 12/ and continues the work in /13 – 15/.

1. We examine a differential game the motion in which is described by the equation

$$x^{\cdot}(t) = f(t,\ x(t),\ u(t),\ v(t)) \tag{1.1}$$

where $u(t) \in P \subset R^p$, $v(t) \in Q \subset R^q$ are the controls of the first and second players, P and Q are compacta and the functions $f(\cdot) : (-\infty,\ \vartheta] \times R^n \times P \times Q \mapsto R^n$ satisfy the usual conditions (see /3,4/). We assume that

$$\min_{u\in P}\max_{v\in Q} s'f(t,x,u,v) = \max_{v\in Q}\min_{u\in P} s'f(t,x,u,v), \qquad ((t,\ x,\ s) \in (-\infty,\ \vartheta] \times R^n \times R^n) \tag{1.2}$$

where $s'f$ is the scalar product of vectors s and f. The differential game's payoff is prescribed by the equality

$$\gamma(x(\cdot)) = \sigma(x(\vartheta)) \tag{1.3}$$

Here $\sigma(\cdot) : R^n \mapsto R$ is a continuous function, ϑ is the fixed game termination instant.

In accord with the formalism in /4/ we identify the strategies of the first and second players with the arbitrary functions $U(\cdot) : (-\infty, \vartheta] \times R^n \mapsto P$, $V(\cdot) : (-\infty, \vartheta] \times R^n \mapsto Q$. The motions generated by such strategies are determined by a limit approach from the corresponding sequences of Euler polygonal lines. The set of motions generated by strategy U and departing from the point $x(t_*) = x_*$ is denoted $X(t_*, x_*, U)$. The sheaf $X(t_*, x_*, V)$ of motions corresponding to strategy V is denoted analogously. It has been established /4/ that a position differential game has a value, i.e.,

$$\min_U \max \gamma(X(t_*, x_*, U)) = \max_V \min \gamma(X(t_*, x_*, V)) = c_0(t_*, x_*)$$
$$\max\ \gamma(X) = \max_{x(\cdot)} \gamma(x(\cdot)),\ \min \gamma(X) = \min_{x(\cdot)} \gamma(x(\cdot)),\ x(\cdot) \in X$$

The quantity $c_0(t_*, x_*)$ is named the value of the differential game, defined for the initial position (t_*, x_*). The function $(t_*, x_*) \mapsto c_0(t_*, x_*)$ is called the value function or the potential of the differential game. The investigation of the properties of the potential and of the methods for computing it is of material interest since by having the potential available we can determine relatively simply the players' optimal strategies (see /2,4/, for example).

Let us note a necessary condition satisfied by the potential of differential game (1.1)–(1.3). Let the function $c(\cdot) : (-\infty, \vartheta] \times R^n \mapsto R$ satisfy in each bounded domain $G \subset (-\infty, \vartheta] \times R^n$ the Lipschitz condition

$$|\ c(t^1,\ x^1) - c(t^2,\ x^2)\ | \leqslant \lambda(G)\ [\,|\ t^1 - t^2\ | + \|\ x^1 + x^2\ \|\,], \qquad \lambda = \text{const} \tag{1.4}$$

The collection of such functions is denoted by the symbol Lip. It can be shown that the potential $c_0(\cdot)$ of game (1.1)–(1.3) belongs to class Lip if the payoff function $\sigma(\cdot)$ satisfies a Lipschitz condition. By the Rademacher theorem /7/ the function $c(\cdot) \in$ Lip is differentiable almost everywhere. The function $c(\cdot)$ is called a generalized solution of the fundamental equation of differential games theory if the equality

$$\frac{\partial c(t_*, x_*)}{\partial t} + \min_{u\in P}\max_{v\in Q} f'(t_*, x_*, u, v)\, \text{grad}_x\, c(t_*, x_*) = 0 \tag{1.5}$$
$$\text{grad}_x\, c(t_*, x_*) = (\partial c(t_*, x_*)/\partial x_1, \ldots, \partial c(t_*, x_*)/\partial x_n)$$

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is fulfilled at each point (t_*, x_*) at which the function $c(\cdot)$ is differentiable. It is well known (see /12/, for instance) that the potential $c_0(\cdot)$ is a generalized solution of the fundamental Eq.(1.5) and satisfies the boundary condition $c_0(\vartheta, x) = \sigma(x)$. However, this necessary condition is not sufficient. Examples can be given in which the fundamental Eq.(1.5) has an infinite set of generalized solutions satisfying the boundary condition $c(\vartheta, x) = \sigma(x)$ $(x \in R^n)$.

In Theorem 2.1 below we have indicated necessary and sufficient conditions which a continuous potential must satisfy. Next, in Theorems 2.2 and 2.3 these conditions are defines more exactly for the cases when the value function belongs to class Lip and is direction-differentiable. The conditions indicated in Theorem 2.3 can be treated as a natural generalization of the fundamental equation.

2. Let us consider the stability properties of functions $(t, x) \mapsto c(t, x)$. These properties, augmented by the boundary condition

$$c(\vartheta, x) = \sigma(x) \quad (x \in R^n) \tag{2.1}$$

form necessary and sufficient conditions which must be satisfied by the potential of differential game (1.1)—(1.3). We remark that stability properties were introduced in /3,4/ for stable bridges in an encounter-evasion game. The stability properties can be defined in different equivalent forms. In particular, we present below an infinitesimal form of the stability property, which leads to a generalization of the fundamental equation.

We introduce some notation. Let $(t, x, u, v) \in (-\infty, \vartheta] \times R^n \times P \times Q$. We set

$$\begin{aligned} F_1(t, x, v) &= \text{co}\,\{f(t, x, u, v): u \in P\} \\ F_2(t, x, u) &= \text{co}\,\{f(t, x, u, v): v \in Q\} \end{aligned} \tag{2.2}$$

where $\text{co}\,A$ is the convex hull of set A. By $X_1(t_*, x_*, v)$ and $X_2(t_*, x_*, u)$ we denote the sets of solutions of the differential inclusions

$$x^{\cdot}(t) \in F_1(t, x(t), v), \quad x^{\cdot}(t) \in F_2(t, x(t), u), \quad (t_* \leqslant t \leqslant \vartheta,\ x(t_*) = x_*) \tag{2.3}$$

respectively. We remark that for any $(t_*, x_*) \in (-\infty, \vartheta] \times R^n$, $u \in P$ and $v \in Q$ the sets $X_1(t_*, x_*, v)$ and $X_2(t_*, x_*, u)$ are nonempty and compact in the space of continuous functions $x(\cdot): [t_*, \vartheta] \mapsto R^n$. For a continuous function $c(\cdot)\ (-\infty, \vartheta] \times R^n \mapsto R$ we define two conditions:

$$(1_u) \qquad \sup_{(t_*, x_*)} \max_t \max_v \min_{x(\cdot)} [c(t, x(t)) - c(t_*, x_*)] \leqslant 0$$

when $(t_*, x_*) \in (-\infty, \vartheta) \times R^n$, $t \in [t_*, \vartheta]$, $v \in Q$, $x(\cdot) \in X_1(t_*, x_*, v)$

$$(1_v) \qquad \inf_{(t_*, x_*)} \min_t \min_u \max_{x(\cdot)} [c(t, x(t)) - c(t_*, x_*)] \geqslant 0$$

when $(t_*, x_*) \in (-\infty, \vartheta) \times R^n$, $t \in [t_*, \vartheta]$, $u \in P$, $x(\cdot) \in X_2(t_*, x_*, u)$. Inequalities (1_u) and (1_v) are called, respectively, the conditions of u-and v-stability of function $c(\cdot)$.

Theorem 2.1. In order that a continuous function $c(\cdot)\ (-\infty, \vartheta] \times R^n \mapsto R$ be the potential of differential game (1.1)—(1.3), it is necessary and sufficient that it satisfy the boundary condition (2.1) and the stability conditions (1_u), (1_v).

Theorem 2.1 follows from the results in /4/; it was proved in /15/ (pp.116-118).

The stability conditions (1_u) and (1_v) for a continuous function $c(\cdot)$ can be determined in various equivalent forms. Consider the following conditions:

$$(2_u) \qquad \sup_{(t_*, x_*)} \max_v \min_{x(\cdot)} \max_t [c(t, x(t)) - c(t_*, x_*)] \leqslant 0$$

$$(3_u) \qquad \sup_{(t_*, x_*)} \sup_v \inf_{x(\cdot)} \varliminf_{\delta \to +0} [c(t_* + \delta, x(t_* + \delta)) - c(t_*, x_*)] \cdot \delta^{-1} \leqslant 0$$

$$(4_u) \qquad \sup_{(t_*, x_*)} \sup_v \inf_{x(\cdot)} \varlimsup_{\delta \to +0} [c(t_* + \delta, x(t_* + \delta)) - c(t_*, x_*)] \cdot \delta^{-1} \leqslant 0$$

$(t_*, x_*) \in (-\infty, \vartheta) \times R^n$, $v \in Q$, $x(\cdot) \in X_1(t_*, x_*, v)$, $t \in [t_*, \vartheta]$

Conditions (2_v), (3_v) and (4_v) are obtained from (2_u), (3_u) and (4_u) by a respective replacement of v by u, of Q by P, of $X_1(t_*, x_*, v)$ by $X_2(t_*, x_*, u)$, and of the symbols max, min, sup, inf, $\leqslant$, by the opposite ones.

Lemma 2.1. Conditions (1_u), (2_u), (3_u) and (4_u) are equivalent.

Lemma 2.2. Conditions (1_v), (2_v), (3_v) and (4_v) are equivalent.

Proof of Lemma 2.1. The implications $(2_u) \Rightarrow (1_u)$, $(2_u) \Rightarrow (4_u) \Rightarrow (3_u)$ are trivial. It can be shown that (2_u) follows from (1_u). Indeed, let $(t_*, x_*) \in (-\infty, \vartheta) \times R^n$, $v \in Q$, k is a positive integer,

$\delta_k = (\vartheta - t_*) k^{-1}$, $\tau_i^k = t_* + i \cdot \delta_k$ $(i = 0, 1, \ldots, k)$. It follows from (1_u) that a motion $x^{(k)}(\cdot) \in X_1(t_*, x_*, v)$ exists satisfying the conditions

$$c(\tau_i^{(k)}, x^{(k)}(\tau_i^{(k)})) \leqslant c(t_*, x_*) \quad (i = 0, 1, \ldots, k) \tag{2.4}$$

Let us consider the sequence $x^{(k)}(\cdot)$ $(k = 1, 2, \ldots)$ of such motions. Since $X_1(t_*, x_*, v)$ is a compactum, it contains the limit element $x^*(\cdot)$ of this sequence. From (2.4) and the continuity of function $c(\cdot)$ it follows that $c(t, x^*(t)) \leqslant c(t_*, x_*)$ for $t_* \leqslant t \leqslant \vartheta$; by the same token we have proved the fulfillment of condition (2_u). It remains to verify that (1_u) follows from (3_u). Assume the contrary. Let condition (3_u) be fulfilled, but let $(t_*, x_*) \in (-\infty, \vartheta) \times R^n$, $v \in Q$, $\alpha > 0$, $t^\circ \in (t_*, \vartheta]$ exist such that

$$\min_{x(\cdot)} c(t^\circ, x(t^\circ)) > c(t_*, x_*) + \alpha, \qquad x(\cdot) \in X_1(t_*, x_*, v) \tag{2.5}$$

Set

$$\tau_*(x(\cdot)) = \max\{t \in [t_*, t^\circ]: c(t, x(t)) \leqslant c(t_*, x_*) + \alpha(t - t_*)(t^\circ - t_*)^{-1}\} \tag{2.6}$$

The functional $\tau_*(\cdot)$ is upper-semicontinuous; therefore, in compactum $X_1(t_*, x_*, v)$ we can choose a motion $x_*(\cdot)$ such that

$$\tau_*(x_*(\cdot)) = \max_{x(\cdot)} \tau_*(x(\cdot)), \qquad x(\cdot) \in X_1(t_*, x_*, v) \tag{2.7}$$

Let $t^* = \tau_*(x_*(\cdot))$, $x^* = x_*(t^*)$; then

$$c(t^*, x^*) = c(t_*, x_*) + \varepsilon(t^* - t_*), \quad \varepsilon = (t^\circ - t_*)^{-1}\alpha \tag{2.8}$$

From (2.5)—(2.7) follows $t^* < t^\circ$. On the strength of (3_u) there exist $\delta \in (0, t^\circ - t^*)$ and motion $x^*(\cdot) \in X_1(t_*, x_*, v)$ such that

$$c(t^* + \delta, x^*(t^* + \delta)) \leqslant c(t_*, x_*) + \varepsilon\delta \tag{2.9}$$

Consider the motion $x(\cdot) = \{x_*(t)$ for $t_* \leqslant t \leqslant t^*$; $x^*(t)$ for $t^* \leqslant t \leqslant \vartheta\}$. This motion is contained in sheaf $X_1(t_*, x_*, v)$ and for it we have the valid inequality

$$c(t^* + \delta, x(t^* + \delta)) \leqslant c(t_*, x_*) + \varepsilon(t^* + \delta - t_*) \tag{2.10}$$

following from (2.8) and (2.9). By the definition of functional $\tau_*(\cdot)$ we obtain $\tau_*(x(\cdot)) \geqslant t^* + \delta$, which contradicts the choice of the number t^* (see (2.7)). Lemma 2.1 is proved. Lemma 2.2 can be proved analogously.

Thus far we have examined continuous functions $c(\cdot)$. Now let $c(\cdot): (-\infty, \vartheta] \times R^n \mapsto R$, $c(\cdot) \in \mathrm{Lip}$. Let $(t_*, x_*) \in (-\infty, \vartheta) \times R^n$, $h \in R^n$. We use the following notation:

$$D^*c(t_*, x_*) \mid (1, h) = \overline{\lim_{\delta \to +0}} [c(t_* + \delta, x_* + h\delta) - c(t_*, x_*)]\delta^{-1}$$
$$D_*c(t_*, x_*) \mid (1, h) = \underline{\lim_{\delta \to +0}} [c(t_* + \delta, x_* + h\delta) - c(t_*, x_*)]\delta^{-1}$$

i.e., here we have introduced the upper and lower derivative numbers of function $c(\cdot)$ at the point (t_*, x_*) in the direction of an $(n+1)$-dimensional vector $(1, h_1, \ldots, h_n)$. If $D^*c(t_*, x_*) \mid (1, h) = D_*c(t_*, x_*) \mid (1, h)$, then at the point (t_*, x_*) the function $c(\cdot)$ has a derivative along the direction of $(1, h)$, which is denoted $Dc(t_*, x_*) \mid (1, h)$. Below we use the following notation:

$$\Delta c(t_*, x_*, x(\cdot), \delta) = c(t_* + \delta, x(t_* + \delta)) - c(t_*, x_*)$$

Lemma 2.3. Let $c(\cdot) \in \mathrm{Lip}$. The inequalities

$$\inf_{x(\cdot)} \underline{\lim_{\delta \to +0}} \Delta c(t_*, x_*, x(\cdot), \delta)\delta^{-1} \leqslant \inf_h D_*c(t_*, x_*) \mid (1, h) \leqslant \inf_{x(\cdot)} \overline{\lim_{\delta \to +0}} \Delta c(t_*, x_*, x(\cdot), \delta) \cdot \delta^{-1} \tag{2.11}$$

$$x(\cdot) \in X_1(t_*, x_*, v), \; h \in F_1(t_*, x_*, v)$$

$$\sup_{x(\cdot)} \underline{\lim_{\delta \to +0}} \Delta c(t_*, x_*, x(\cdot), \delta)\delta^{-1} \leqslant \sup_h D^*c(t_*, x_*) \mid (1, h) \leqslant \sup_{x(\cdot)} \overline{\lim_{\delta \to +0}} \Delta c(t_*, x_*, x(\cdot), \delta)\delta^{-1} \tag{2.12}$$

$$x(\cdot) \in X_2(t_*, x_*, u), \; h \in F_2(t_*, x_*, u)$$

are valid for any $(t_*, x_*) \in (-\infty, \vartheta) \times R^n$, $v \in Q$, and $u \in P$.

Proof. Let us prove the first of inequalities (2.11). Let $(t_*, x_*) \in (-\infty, \vartheta) \times R^n$, $v \in Q$, $\varepsilon > 0$. We choose $h_* \in F_1(t_*, x_*, v)$ and a sequence δ_k $(k = 1, 2, \ldots)$, $\delta_k \to +0$ as $k \to \infty$, such that

$$\lim_{k \to \infty} [c(t_* + \delta_k, x_* + \delta_k h_*) - c(t_*, x_*)]\delta_k^{-1} = D_*c(t_*, x_*) \mid (1, h_*) \leqslant \inf_h D_*c(t_*, x_*) \mid (1, h) + \varepsilon, \qquad h \in F_1(t_*, x_*, v) \tag{2.13}$$

From the continuity of the multivalued mapping $(t,x) \mapsto F_1(t, x, v)$ of (2.2) follows the existence of a solution $x_*(\cdot) \in X_1(t_*, x_*, v)$ (see (2.3)) such that

$$x_*(t_* + \delta_k) = x_* + \delta_k h_* + g_k \delta_k, \qquad (\|g_k\| \to 0,\ k \to \infty) \tag{2.14}$$

We set

$$r_k = [c(t_* + \delta_k, x_*(t_* + \delta_k)) - c(t_* + \delta_k, x_* + \delta_k h_*)]\,\delta_k^{-1} \tag{2.15}$$

From (2.14), by virtue of the condition $c(\cdot) \in \mathrm{Lip}$, we obtain

$$r_k \to 0, \qquad k \to \infty \tag{2.16}$$

From (2.13)—(2.16) follows

$$\begin{aligned}&\inf_{x(\cdot)} \varliminf_{\delta \to +0} \Delta c(t_*, x_*, x(\cdot), \delta)\,\delta^{-1} \leqslant \lim_{k\to\infty} \Delta c(t_*, x_*, x_*(\cdot), \delta_k)\,\delta_k^{-1} = \\ &\lim_{k\to\infty} [c(t_* + \delta_k, x_* + h_*\delta_k) - c(t_*, x_*)]\,\delta_k^{-1} \leqslant \\ &\inf_h D_* c(t_*, x_*) \mid (1, h) + \varepsilon, \qquad h \in F_1(t_*, x_*, v)\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary here, we obtain the first one of inequalities (2.11).

Now we choose a motion $x_*(\cdot) \in X_1(t_*, x_*, v)$ and a sequence $\delta_k > 0$ $(k = 1, 2, \ldots,\ \delta_k \to 0$ as $k \to \infty)$ such that

$$\inf_{x(\cdot)} \varlimsup_{\delta \to +0} \Delta c(t_*, x_*, x(\cdot), \delta)\,\delta^{-1} \geqslant \lim_{k\to\infty} \Delta c(t_*, x_*, x_*(\cdot), \delta_k)\,\delta_k^{-1} - \varepsilon \tag{2.17}$$

From the continuity of mapping $(t, x) \mapsto F_1(t, x, v)$ it follows that

$$x_*(t_* + \delta_k) = x_* + h_k\delta_k + g_k^*\delta_k \qquad (h_k \in F_1(t_*, x_*, v),\ \|g_k^*\| \to 0,\ k \to \infty)$$

Set $F_1(t_*, x_*, v)$ is compact and, therefore, from the sequence h_k $(k = 1, 2, \ldots)$ we can choose a subsequence converging to the limit $h_* \in F_1(t_*, x_*, v)$. For simplicity of notation we assume that $h_k \to h_* \in F_1(t_*, x_*, v)$ as $k \to \infty$. Then we arrive at (2.14) wherein $g_k = g_k^* + h_k - h_*$. From (2.14) — (2.17) follows

$$\begin{aligned}&\inf_{x(\cdot)} \varlimsup_{\delta \to +0} \Delta c(t_*, x_*, x(\cdot), \delta)\,\delta^{-1} + \varepsilon \geqslant \lim_{k\to\infty} \Delta c(t_*, x_*, x_*(\cdot), \delta_k)\cdot\delta_k^{-1} \geqslant \\ &\inf_h D_* c(t_*, x_*) \mid (1, h), \qquad h \in F_1(t_*, x_*, v), \qquad x(\cdot) \in X_1(t_*, x_*, v)\end{aligned}$$

Here again $\varepsilon > 0$ is arbitrary, and, therefore, we obtain the second of inequalities (2.11). Inequalities (2.12) are proved by the same scheme.

From Theorem 2.1 and Lemmas 2.1 – 2.3 follows

Theorem 2.2. In order that a function $c(\cdot)$ belonging to class Lip be the potential of differential game (1.1)— (1.3), it is necessary and sufficient that it satisfy the boundary condition (2.1) and that the inequalities

$$\begin{aligned}&\sup_{v \in Q}\ \inf_{h \in F_1(t_*, x_*, v)} D_* c(t_*, x_*) \mid (1, h) \leqslant 0, \\ &\inf_{u \in P}\ \sup_{h \in F_2(t_*, x_*, u)} D^* c(t_*, x_*) \mid (1, h) \geqslant 0\end{aligned}$$

be fulfilled at each position $(t_*, x_*) \in (-\infty, \vartheta) \times R^n$.

In /13/ it was remarked that in the formulation of the necessary and sufficient conditions for the potential $c(\cdot)$ the upper derivative numbers can be replaced by the lower derivative numbers and vice versa; this has turned out to be unjustified and to date we have been unalbe to prove it or to refute it.

By Dif we denote the collection of functions $c(\cdot)$ which at any point $(t_*, x_*) \in (-\infty, \vartheta) \times R^n$ have a derivative along any direction $(1, h)$, $h \in R^n$. For a function $c(\cdot) \in \mathrm{Lip} \cap \mathrm{Dif}$ we can refine Theorem 2.2 as follows.

Theorem 2.3. In order that a function $c(\cdot)$ belonging to class $\mathrm{Lip} \cap \mathrm{Dif}$ be the potential of differential game (1.1)— (1.3), it is necessary and sufficient that it satisfy the boundary condition (2.1) and that the inequalities

$$\begin{aligned}&\max_{v \in Q}\ \min_{h \in F_1(t_*, x_*, v)} D\,c(t_*, x_*) \mid (1, h) \leqslant 0 \\ &\min_{u \in P}\ \max_{h \in F_2(t_* x_*, u)} Dc(t_*, x_*) \mid (1, h) \geqslant 0\end{aligned} \tag{2.18}$$

be fulfilled at each position $(t_*, x_*) \in (-\infty, \vartheta) \times R^n$.

We note here that the maximum and the minimum are reached here, since for a function $c(\cdot)$

of class Lip $\cap$ Dif the mapping $h \mapsto D\, c\,(t_*,\, x_*)\mid(1,\, h)$ satisfies a Lipschitz condition, the multi-valued mappings $v \mapsto F_1\,(t_*,\, x_*,\, v)$ and $u \mapsto F_2\,(t_*,\, x_*,\, u)$ are continuous, and the sets $F_1\,(t_*,\, x_*,\, v)$, $F_2\,(t_*,\, x_*,\, u)$, P and Q are compact.

Notes. 1°. Theorems 2.1–2.3 have been formulated for a potential defined in the whole position space $(-\infty,\, \vartheta] \times R^n$. It can be shown that they remain valid in any domain of the form

$$\{(t,\, x):\ t_{00} \leqslant t \leqslant \vartheta,\ \alpha < c\,(t,\, x) < \beta\}$$

2°. If function $c\,(\cdot)$ is differentiable at point $(t_*,\, x_*)$ then

$$Dc\,(t_*,\, x_*)\mid(1,\, h) = \partial c\,(t_*,\, x_*)/\partial t + h' \operatorname{grad}_x c\,(t_*,\, x_*)$$

Therefore, equality (1.5) follows from (1.2) and (2.18), i.e., we obtain the necessary condition stated at the end of Sect.1 for the potential.

3°. The results obtained above carry over to the case of differential games in which condition (1.2) can be violated. A transition from the payoff functions (1.3) to payoffs of other types is possible as well; for example, $\gamma\,(x\,(\cdot)) = \min_t \omega\,(t,\, x\,(t))$ for $t_* \leqslant t \leqslant \vartheta$. The corresponding results have been formulated in /13/.

3. Let us determine certain classes of direction-differentiable functions. Let I and J be finite sets, $\varphi_{ij}\,(\cdot)$: $(-\infty,\, \vartheta] \times R^n \mapsto R$ $(i \in I,\ j \in J)$ be continuously differentiable functions. We define a piecewise-smooth function

$$(t,\, x) \mapsto c\,(t,\, x) = \min_{i\in I}\ \max_{j\in J}\ \varphi_{ij}\,(t,\, x) \tag{3.1}$$

The piecewise-smooth function if direction-differentiable and the formula

$$\begin{aligned} &Dc\,(t_*,x_*)\,|\,(1,h) = \min_i \max_j\,[\partial\varphi_{ij}\,(t_*,x_*)/\partial t + h' \operatorname{grad}_x \varphi_{ij}\,(t_*,x_*)],\\ &i \in I_0\,(t_*,x_*),\quad j \in J_0\,(t_*,x_*,i)\\ &I_0\,(t_*,x_*) = \{i_0 \in I:\ \max_{j\in J} \varphi_{i_0 j}\,(t_*,x_*) = c\,(t_*,x_*)\}\\ &J_0\,(t_*,x_*,i) = \{j_0 \in J:\ \max_{j\in J} \varphi_{ij}\,(t_*,x_*) = \varphi_{ij_0}(t_*,x_*)\} \end{aligned} \tag{3.2}$$

is valid /1/. We introduce one more type of direction-differentiable functions $c\,(\cdot)$. Let S be a nonempty set. Let Π be some set of scalar functions $\pi\,(\cdot)$: $S \mapsto R$. A functional mix: $\Pi \mapsto R$ is defined on set Π. The value that the operation mix associates with the function $\pi\,(\cdot)$ is denoted $\operatorname{mix} \pi\,(s)$. We assume that the functional mix has the following properties. If $\pi\,(\cdot) \in \Pi$, $r \in R$, then the functions $s \mapsto \pi\,(s) + r$ and $s \mapsto |\,r\,|\,\pi\,(s)$ as well belong to Π and the equalities

$$\operatorname*{mix}_s\,(r + \pi\,(s)) = r + \operatorname*{mix}_s \pi\,(s),\quad \operatorname*{mix}_s\ |\,r\,|\ \pi\,(s) = |\,r\,| \operatorname*{mix}_s \pi\,(s)$$

are valid. If $\pi_i\,(\cdot) \in \Pi$ $(i = 1,\, 2)$ and $\pi_1\,(s) \leqslant \pi_2\,(s)$ $(s \leqslant S)$, then

$$\operatorname*{mix}_s \pi_1\,(s) \leqslant \operatorname*{mix}_s \pi_2\,(s)$$

Let $(t,\, x) \mapsto c\,(t,\, x)$ be a continuous function. We say that function $c\,(\cdot)$ is regular at the point $(t_*,\, x_*) \in (-\infty,\, \vartheta) \times R^n$ if the relations

$$\begin{aligned} &c\,(t,\ x) = \operatorname{mix}_s\ \varphi\,(t,\ x,\ s)\ ((t,\ x) \in Q_\alpha\,(t_*,\ x_*))\\ &c\,(t_*,\ x_*) = \varphi\,(t_*,\ x_*,\ s) \sim (s \in S)\\ &O_\alpha\,(t_*,\ x_*) = \{(t,\ x):\ t_* \leqslant t \leqslant t_* + \alpha,\ \|\,x - x_*\,\| \leqslant \alpha\} \end{aligned} \tag{3.3}$$

are fulfilled for some $\alpha \in (0,\, \vartheta - t_*)$. Here the function $\varphi\,(\cdot)$: $O_\alpha\,(t_*,\, x_*) \times S \mapsto R$ satisfies the following conditions: $\operatorname{grad}_x \varphi\,(t,\, x,\, s)$ and the right derivative $\partial\varphi\,(t_*,\, x_*,\, s)/\partial t$ exist for any $s \in S$; furthermore,

$$\begin{aligned} &\lim_{(t,\,x)\to(t_*,\,x_*)}\ \sup_{s\in S} |\,\varphi\,(t,x,s) - \varphi\,(t_*,x_*,s) - (t - t_*)\,\partial\varphi\,(t_*,x_*,s)/\partial t\ -\\ &\quad (x - x_*)' \operatorname{grad}_x \varphi\,(t_*,x_*,s)\,|\,(|\,t - t_*\,| + \|\,x - x_*\,\|)^{-1} = 0\\ &\sup_{s\in S}\ |\ \partial\varphi\,(t_*,\ x_*,\ s)/\partial x_i\ | < \infty\quad (i = 1,\ .\ .\ .,\ n) \end{aligned}$$

In (3.3) mix is an operation of the form indicated. It is assumed that

$$\begin{aligned} &\varphi\,(t,\ x,\ \cdot) \in \Pi,\ [\partial\varphi\,(t_*,\ x_*,\ \cdot)/dt + h' \operatorname{grad}_x \varphi\,(t_*,\ x_*,\ \cdot)] \in \Pi\\ &\forall\ (t,\ x) \in O_\alpha\,(t_*,\ x_*),\quad h \in R^n \end{aligned}$$

A function $c\,(\cdot)$ of form (3.3) is differentiable at point $(t_*,\, x_*)$ along the direction $(1,\, h)$, and

$$Dc(t_*, x_*) \mid (1, h) = \operatorname*{mix}_{s} [\partial\varphi(t_*, x_*, s)/\partial t + h' \operatorname*{grad}_{x} \varphi(t_*, x_*, s)] \tag{3.4}$$

We note that the above-defined properties of operation mix coincide completely with the properties, indicated in /6/, of the functional $\mathrm{Val}(\cdot): \sigma \mapsto \mathrm{Val}(\sigma)$, where $\mathrm{Val}(\sigma)$ is the value of a differential game with payoff σ (for a fixed initial position).

Using the expressions for derivatives (3.2) and (3.4) we can make Theorem 2.3 concrete for piecewise-smooth functions (3.1) and for functions $c(\cdot)$ regular in the domain $(-\infty, \vartheta) \times R^n$.

In conclusion we present a corollary to Theorem 2.3 for an optimal control problem. In this problem we are required, by choosing a programmed control $u(\cdot)$: $[t_*, \vartheta] \mapsto P$, to minimize the value of functional $\sigma(x(\vartheta, t_*, x_*, u(\cdot)))$, where $x(\cdot, t_*, x_*, u(\cdot))$ is a motion of the controlled system

$$x^{\cdot}(t) = f(t, x(t), u(t)),\ x(t_*) = x_*$$

For simplicity we assume that the set $F(t, x) = \{f(t, x, u): u \in P\}$ is convex for any point (t, x). The quantity $\rho_0(t_*, x_*) = \min_{u(\cdot)} \sigma(x(\vartheta, t_*, x_*, u(\cdot)))$ is called the optimal result in the control problem and the function $(t_*, x_*) \mapsto \rho_0(t_*, x_*)$ is called the potential in the control problem. The potential $\rho_0(\cdot)$ coincides with the potential $c_0(\cdot)$ of the differential game in which $f(t, x, u, v) = f(t, x, u) + v$, $v \in Q = \{0\} \subset R^n$, i.e., the second player is in fact absent. Assume that the functions $\sigma(\cdot)$ and $f(\cdot)$ are differentiable. Then, according to /5/, the potential $\rho_0(\cdot)$ is direction-differentiable. The next theorem is a corollary of Theorem 2.3.

Theorem 3.1. In order that the function $(t, x) \mapsto \rho(t, x)$ be the potential in the control problem being examined, it is necessary and sufficient that the relations

$$\min_{u \in P} D\rho(t, x) \mid (1, f(t, x, u)) = 0 \tag{3.5}$$

$$\rho(\vartheta, x) = \sigma(x),\ (t \in (-\infty, \vartheta),\ x \in R^n) \tag{3.6}$$

be fulfilled.

We remark that a necessary condition for potential $\rho_0(\cdot)$, close to equality (3.5), was obtained in /5/.

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